NAVAL
POSTGRADUATE
SCHOOL

Search Theory Tutorial

Professor Alan Washburn
Operations Research Department
US Naval Postgraduate School

December, 2009

taken from chapter 7 of Washburn/Kress reference

World War Two origins

- Long range, non-intuitive sensors like radar and sonar had just been invented
- Antisubmarine warfare (ASW) was crucial throughout the war
  - over 600 ships were lost off the East coast of the USA in the first half of 1942
  - this disaster led to the creation of ASWORG, predecessor of OEG and the current Center for Naval Analyses
  - the main difficulty in ASW is finding the submarine
- Concepts such as lateral range, sweepwidth, random search, dynamic enhancement, detection rate …
  - all will be introduced in this tutorial
  - original source is OEG report 56, issued in 1946

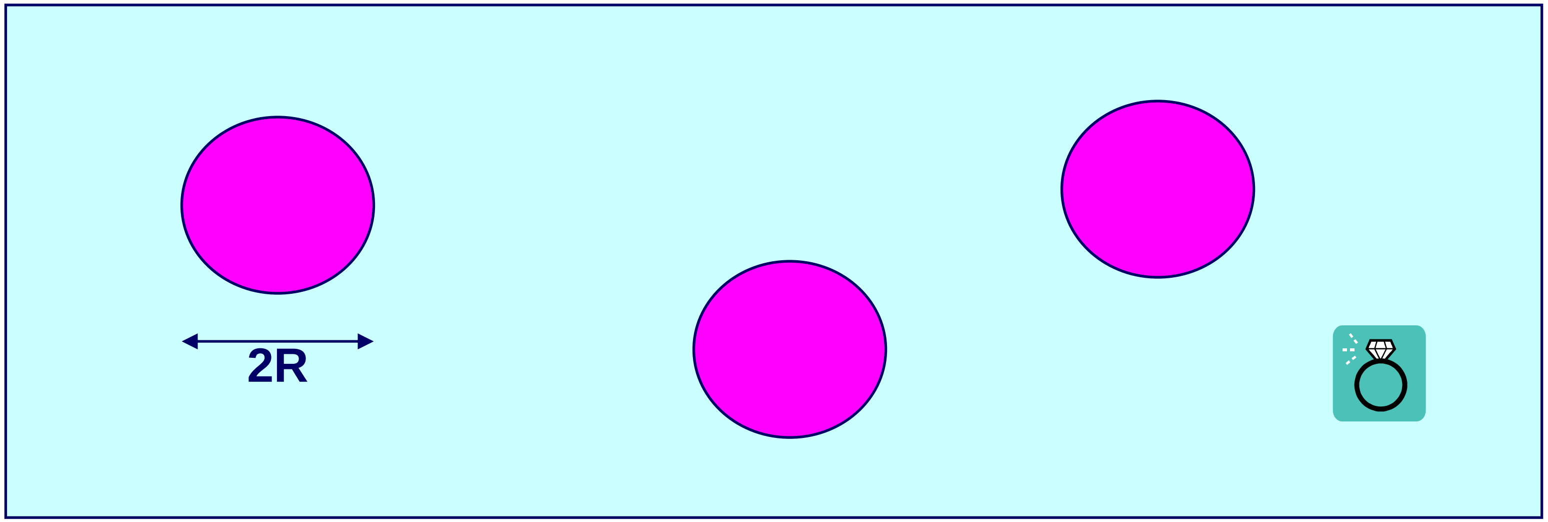
Sensors and targets

- Eyes
  - and other electromagnetic sensors like radar
- Ears
  - and other acoustic sensors like sonar
- All sensors require *proximity* for detection
  - detection radius (R) in the cookie-cutter case
- All targets are point targets (no partial detections)
- We do not deal with internet searches, searches for the meaning of life, etc.

Discrete search

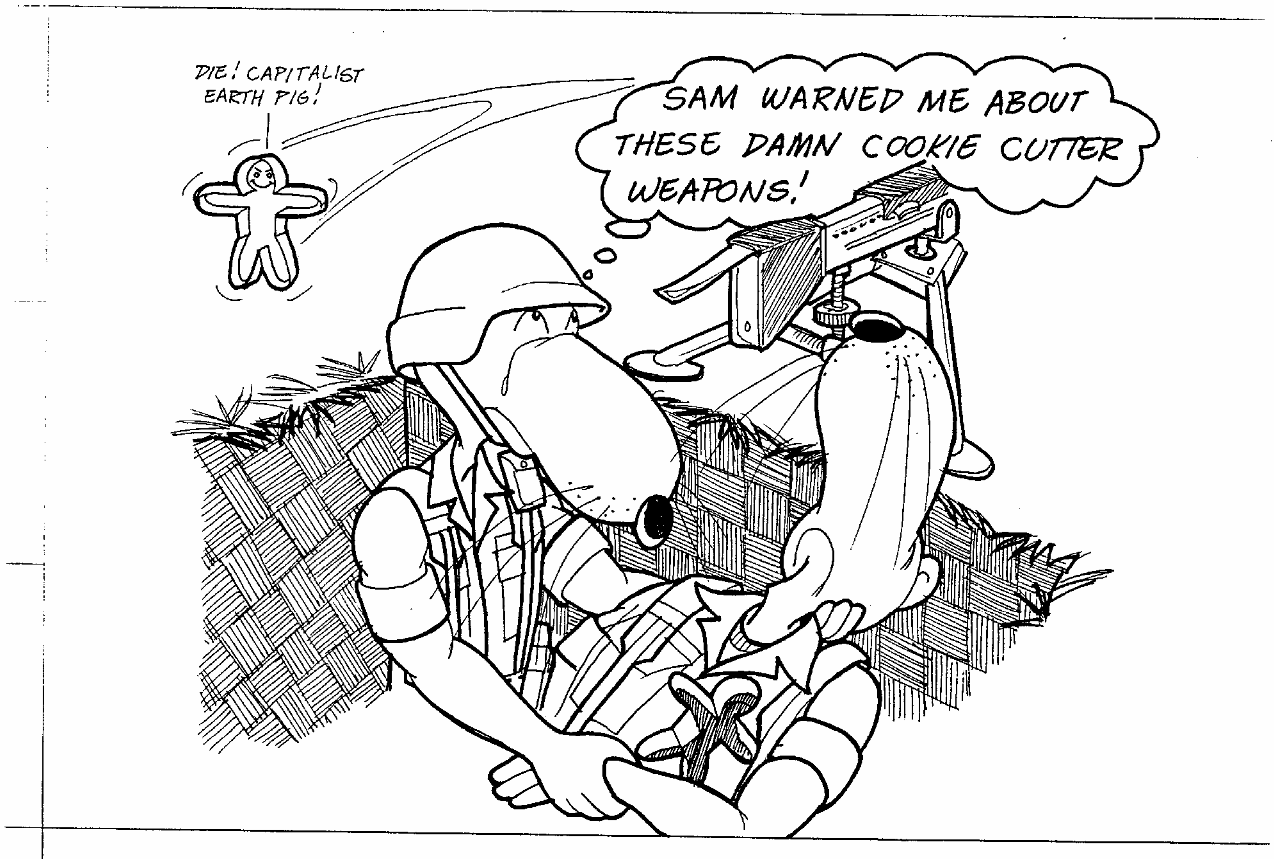
NAVAL POSTGRADUATE SCHOOL

- A sequence of attempts at detection
  - sonobuoys, dipping sonar
- Discrete search theory has much in common with firing theory
  - detection (search) ⇔ kill (firing)
- Picture shows “cookie-cutter” looks/shots with radius R



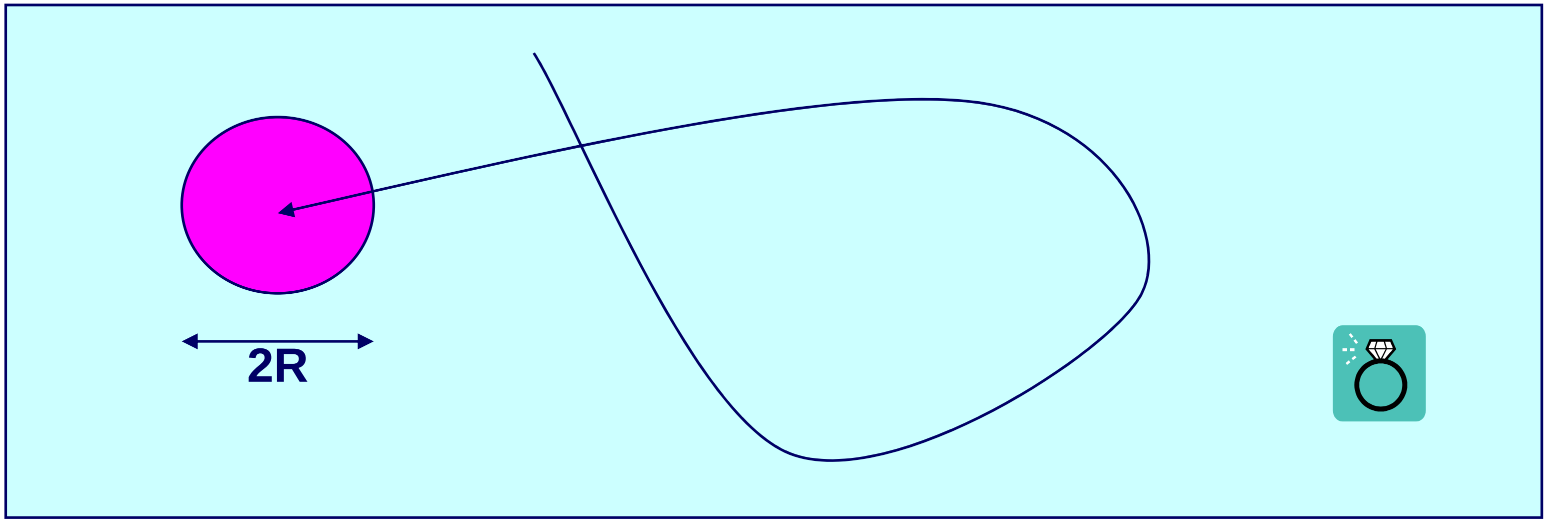
NAVAL POSTGRADUATE SCHOOL

1991 cartoon by Gus Stafford (student in NPS OA curriculum)



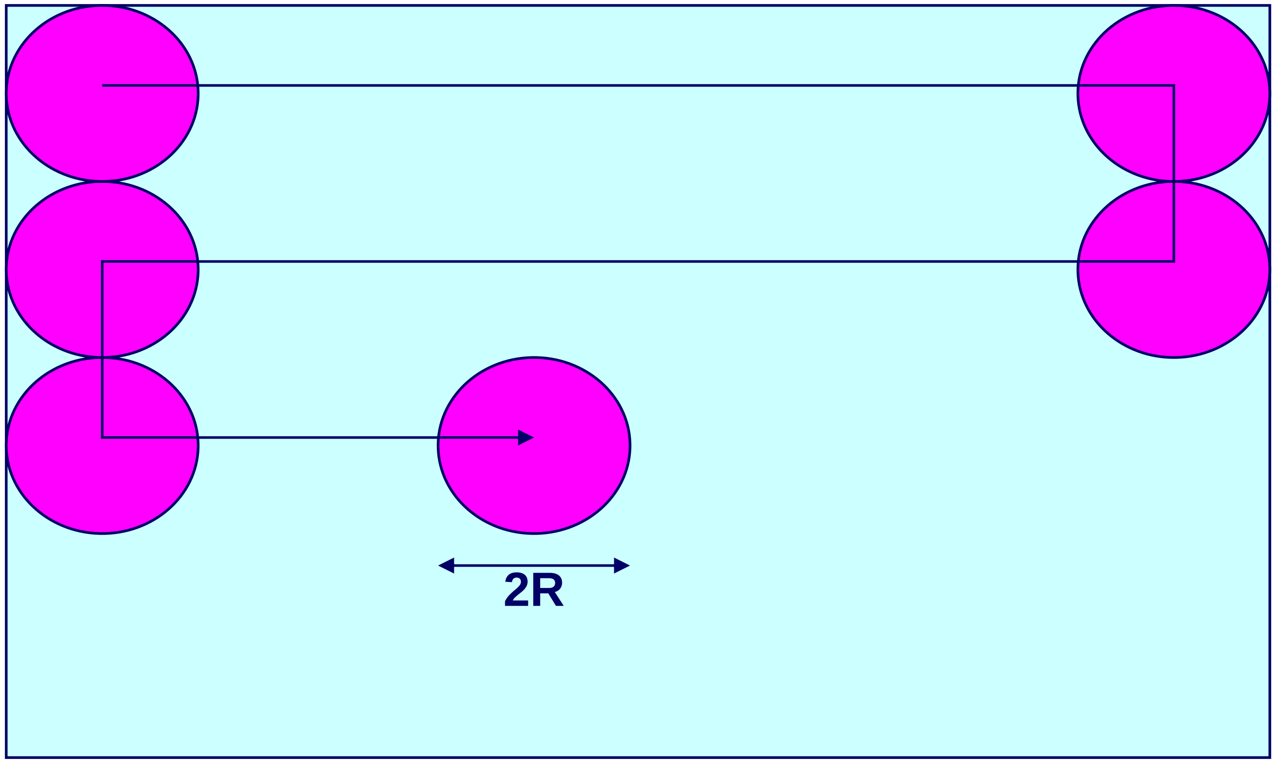
Continuous search

- The sensor moves continuously in time
  - – eyeballs, ears, IR detectors, radars, …
  - – our main focus in this lecture



Mowing the lawn

- Back and forth with no gaps and no overlap



NAVAL POSTGRADUATE SCHOOL

Exhaustive search

- Search within A at constant speed (V) and constant detection radius (R) with no wasted effort, until detection happens, more or less as in mowing the lawn
  - sweepwidth W=2R
  - rate of covering area =V W
  - time to cover area A is A/(V W)
- Let random variable T= time to detection
  - T is uniform in [0, A/(V W)]
  - E(T)=A/(2V W)
- This is an optimistic view of the search problem

Example (exhaustive eyeball search for a lifeboat)

- V=100 kt
- W= 0.2 nm
- A= 100 nm^2
  - A/(V W)=5 hr
  - E(T)=mean time to detection =2.5 hr
  - P(T<4 hr)=0.8

What goes wrong?

- Not sure about A
- Not sure about cookie-cutter assumption
  - target = ?
  - environment = ?
  - sensor = ?
- Can't navigate perfectly
- Target moves
- Circles don't pack well
- Coffee breaks, attention gaps, etc.

NAVAL POSTGRADUATE SCHOOL

Random search

- A skeptical reaction to the optimism of the exhaustive search assumptions
- Assume that the probability of detecting the target in a *small* interval of time of length Δ is V WΔ/A
  - *independently* for all small intervals
  - exhaustive search permits Δ to be small or large, and does not assume independence
- In time t, there are n=t/Δ of these small intervals
- By independence, $P(T>t)=(1- V\,W\Delta/A)^n \to \exp(-\lambda t)$
  - λ=V W/A= "detection rate"
  - T is an exponential random variable
  - E(T)=1/ λ (twice as large as in exhaustive search)
- In effect, the total area covered by time t is chopped up into little bits of confetti and scattered over A, instead of being organized into a single sheet covering V Wt as in exhaustive search
- Exhaustive search is organized, random search is disorganized

NAVAL POSTGRADUATE SCHOOL

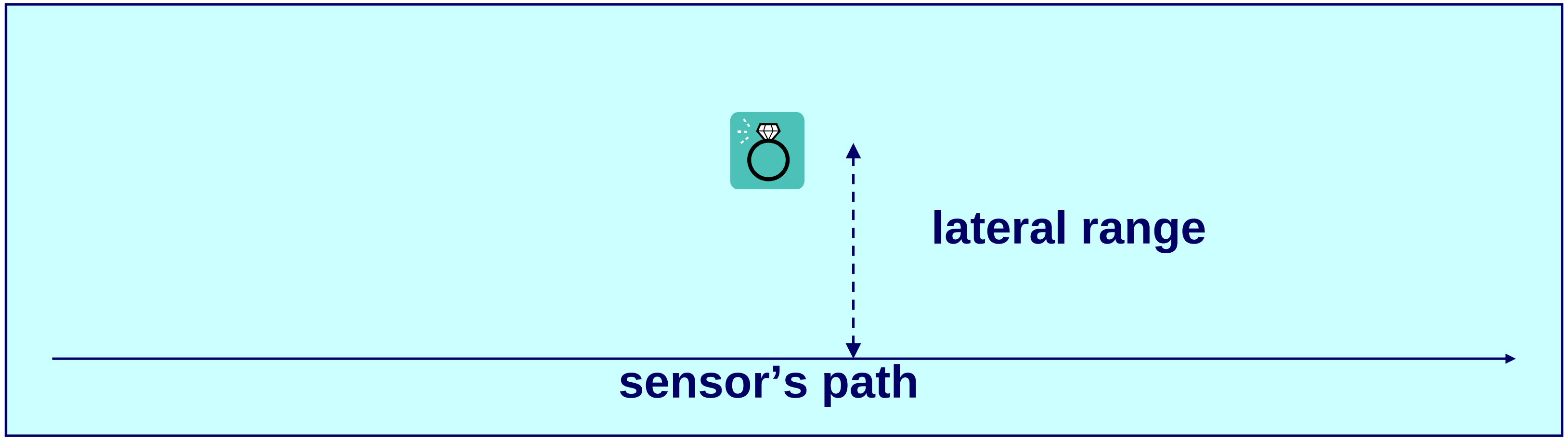
The random search formula

- PD(t)= detection probability by time t
- PD(t)= P(T≤t) = 1- exp(- λt), where λ =VW/A
  - this is the most famous formula in search theory
  - developed in World War II in reaction to the optimism of exhaustive search
  - exponential random variables also have analytic advantages
- random search is not a goal, but rather a skeptical prediction of the ultimate effect of trying to cover an area uniformly

NAVAL POSTGRADUATE SCHOOL

Lateral range

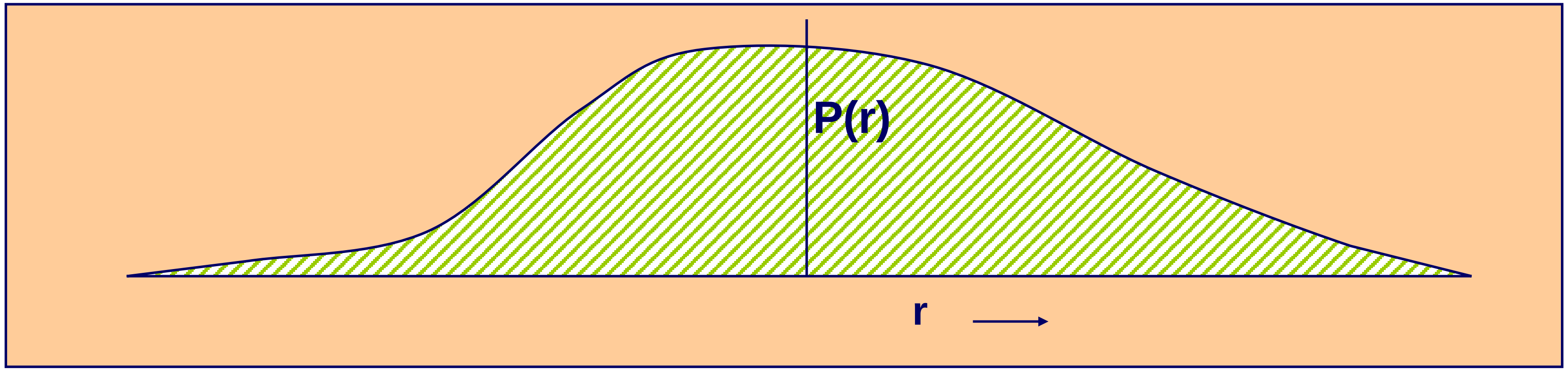
- If the sensor is not cookie-cutter, then how can we describe it?
  - many sensors do not have a definite detection range R, as previously assumed
- We cannot begin by measuring detection probability as a function of distance from the target
  - the experiment is not clear – how much *time* do we spend at that distance?
- Instead, imagine that the sensor moves past the target at constant speed on a doubly infinite straight line, a "pass" at the target
  - the distance at the closest point of approach is the lateral range



NAVAL POSTGRADUATE SCHOOL

Lateral range curves

- As the sensor moves past the target, it may or may not detect it
- Let P(r) be the detection probability when the lateral range of a pass is r
  - the detection might be before or after the closest point of approach – we don't distinguish
- A graph of P(r) versus r is a "lateral range curve"
  - the US Coast Guard measures these for detection of life rafts by eyeballs
  - the US Navy measures these for detection of ships by mines
- The area under the lateral range curve (shaded) is the sweep width W
  - this generalizes the cookie-cutter definition

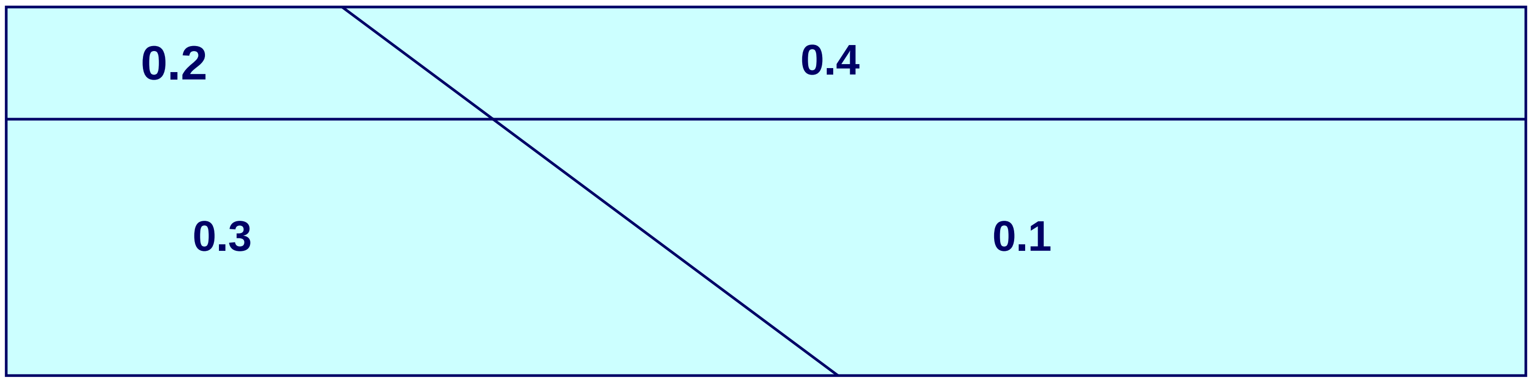


Uses and misuses of lateral range curves

- The most common use is to find the sweep width and plug it into the random search formula
- One can also consider a comb of passes with lateral ranges r_1, r_2, …, r_n, assuming that all passes are independent
  - comb detection probability=1-(1-P(r_1)(1-P(r_2)…(1-P(r_n))
- Successful applications always involve multiple passes, with little turning on each pass
- There are many search problems where "passes", and therefore lateral range curves, don't make sense
  - searching a 10 nm by 10 nm area with a sensor whose sweep width is 10 nm
  - circling the target at constant range r for three hours
    - given only the lateral range curve, we cannot determine the detection probability, and it is certainly not P(r)
  - it is tempting to use them anyway

Cell partitioning

- The target's location may not be uniformly distributed over the area A, or search conditions may not be the same everywhere
  - spatial restrictions on visibility and sensor speed
  - topography/bathymetry can be important
  - historical experience with where targets are to be found
- Solution: partition A into cells where the uniformity assumption is reasonable
  - p_i= probability that the target is in cell i
  - 1978 search for USS Scorpion involved about 100 cells



Two solvable problems

- Given a fixed amount of time for random searching, how should that time be divided among the cells of a partition?
  - a nonlinear optimization problem
  - the greedy algorithm will produce an optimal solution
    - always put the next increment of search effort in the cell where detection is most likely
- Given that random search has failed up to time t, what is the revised distribution of the target's position?
  - a handover question if multiple searchers are involved
  - answerable with Bayes theorem

NAVAL POSTGRADUATE SCHOOL

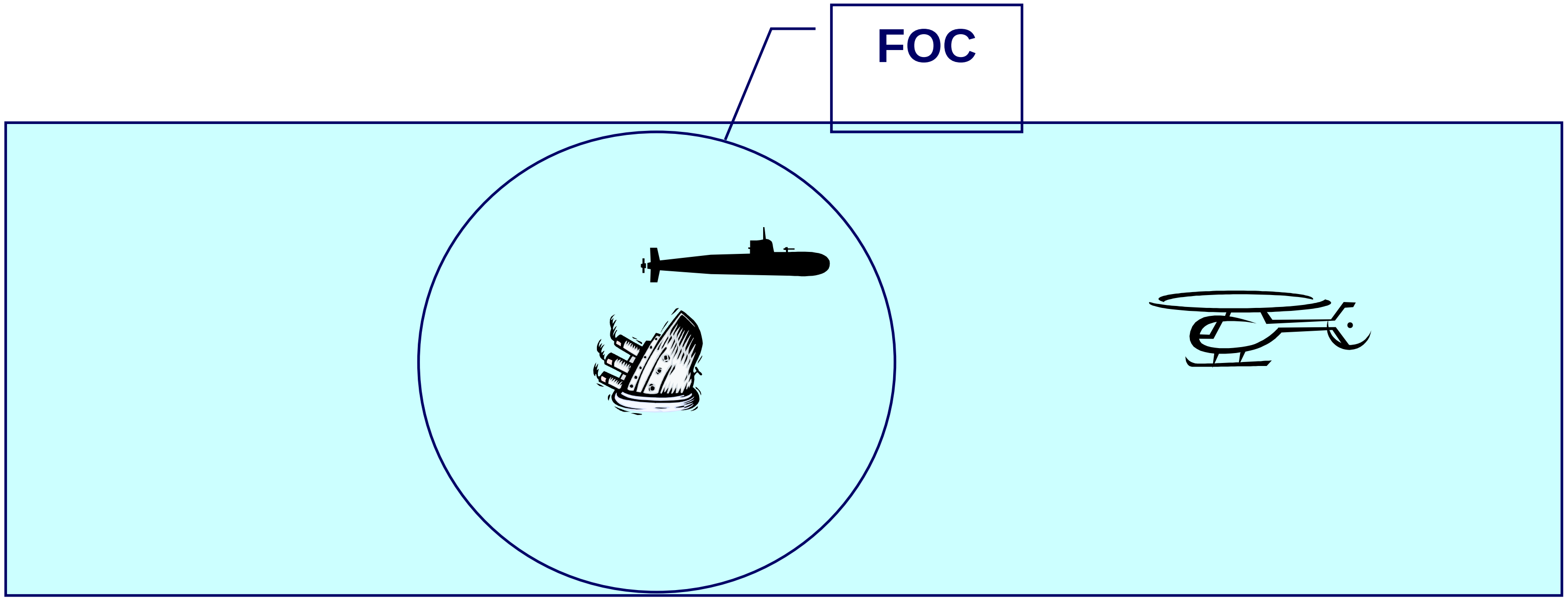
Generalizations of random search

- In random search, we assume that the detection probability in a small time Δ is $\lambda\Delta$, independently in all small intervals, where λ=V W/A
- The generalization is that λ can depend on time
  - let λ(u) be the detection rate at time u
- λt in the random search formula is replaced by the integral of λ(u) between the limits of 0 and t
  - (detections are a nonhomogeneous Poisson process)

$$PD(t) = 1 - \exp(-\int_0^t \lambda(u)du) = 1 - \exp(-z(t))$$

The flaming datum problem

- A submarine torpedoes a ship (the "flaming datum")
  - Or any other incident known to both sides
- ASW forces arrive after a delay and begin searching for the submarine
- Submarine moves within the expanding farthest-on-circle (FOC) as search proceeds
- Detection of sub made quickly, if at all



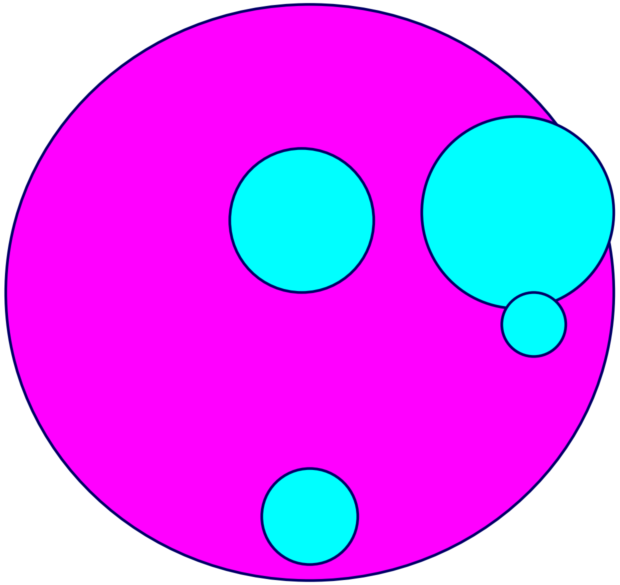
Flaming datum analysis

- U=speed of submarine (FOC radius at time u is Uu)
- ρ=area coverage rate (V times W, say)
  - dipping sonars also have an area coverage rate
- λ(u)= ρ /{π(Uu)^2}=detection rate at time u
- τ=time late
- z(t)=average number of detections between τ and t
- PD(t)=1-exp(-z(t))

$$z(t) = \int_\tau^t \frac{\rho}{\pi (Uu)^2} du = \frac{\rho}{\pi U^2}(1/\tau - 1/t)$$

Dipping sonar animation

Animation with fixed FOC



NAVAL POSTGRADUATE SCHOOL

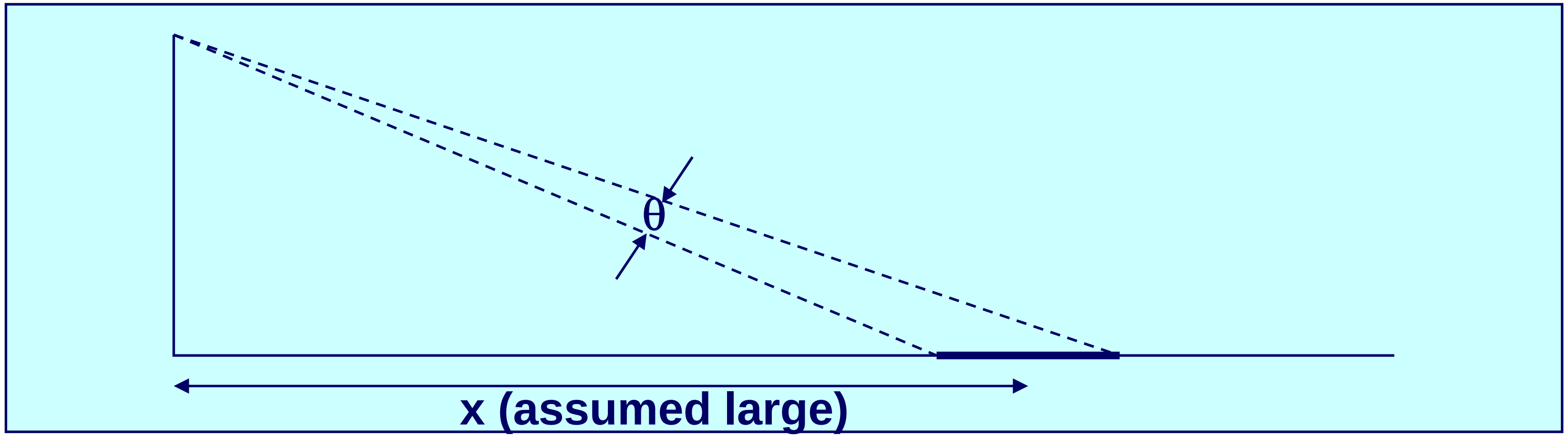
Diesel Submarines

- **Widespread, increasingly lethal weapon systems**
- **Original source of Flaming Datum problem**
- **Energy constraint**
  - **finite battery capacity prevents high speeds, except briefly**
  - **the constraint on speed must be replaced or augmented by a constraint on energy use**
  - **see Soto (1989) or Hohzaki and Washburn (2001)**

NAVAL POSTGRADUATE SCHOOL

Inverse Cube Law

- The solid angle subtended by a wake at distance x is approximately proportional to x^{-3} (the planar angle θ shown below is proportional to x^{-2})
- λ(u) is proportional to the solid angle, which changes with u as the observer passes the target
- Theory leads to analytic formula for the lateral range curve, and also for the detection probability of a comb of passes
  - detection probability is between random and exhaustive
  - used as a compromise even when the object of detection is not a wake



NAVAL POSTGRADUATE SCHOOL

IAMSAR Manual

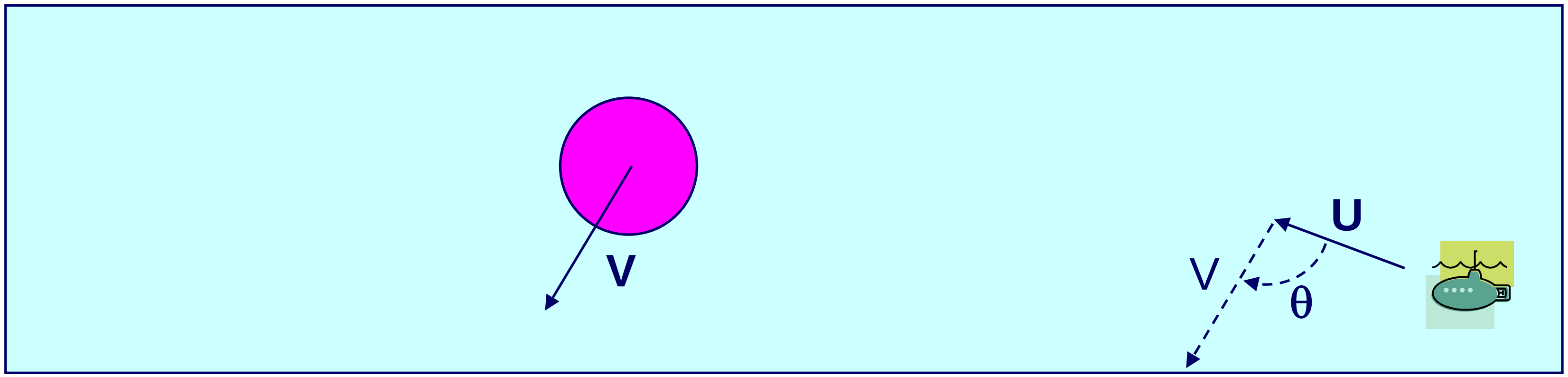
- **The *International Aeronautical and Maritime Search and Rescue Manual* includes instructions for predicting detection probability**
  - first, determine the sweep width W from a table
  - if search conditions are “ideal”, use the inverse cube law
  - if search conditions are “poor”, use the random search formula
  - the IAMSAR manual never uses the exhaustive search formula

NAVAL POSTGRADUATE SCHOOL

Search for a Moving Target

- A target with top speed U is confined within an area A
- A searcher moves with speed V in a random search for the target
- What does U have to do with the detection probability?
- The average relative speed is V_r, the “dynamically enhanced” speed that should replace V in the random search formula
  - V_r is symmetric in V and U, and larger than either one

$$V_r = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{V^2+U^2-2UV\cos(\theta)}\,d\theta$$



•Questions: Why should an evasive target move at all?

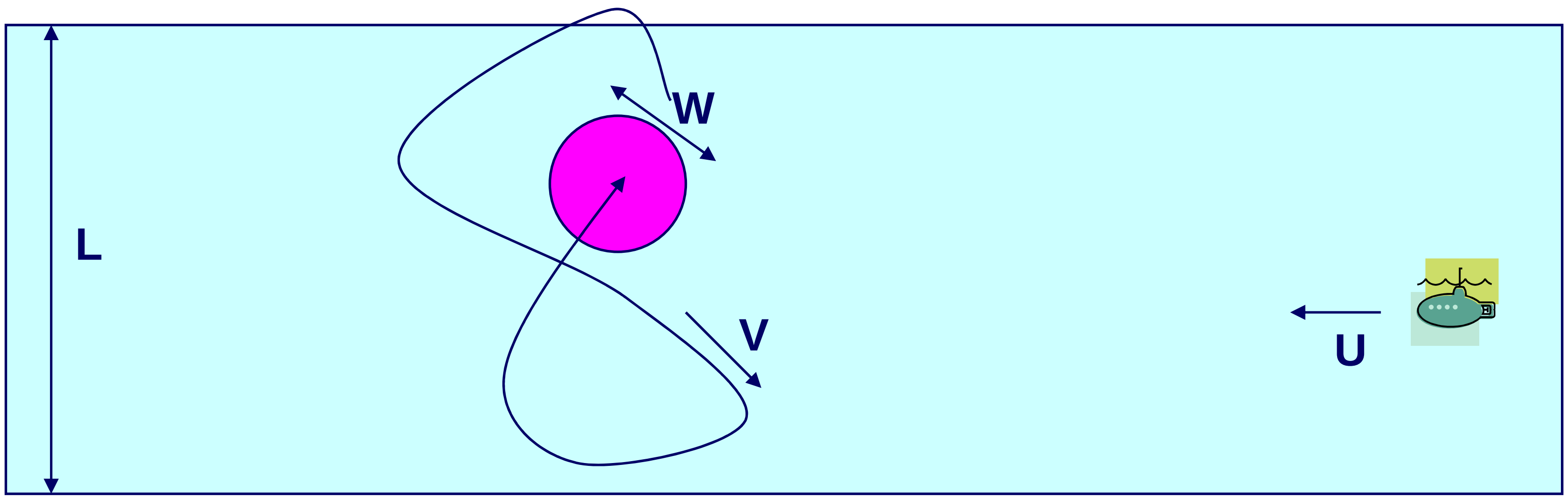
•What should an evasive target do if he can see the searcher?

NAVAL POSTGRADUATE SCHOOL

Barrier search

- A transitor moves at speed U through a channel of width L
- A searcher moves at speed V around a fixed path of his choice
- What closed path should the searcher follow, and what is the resultant detection probability?
- The approximate answers are known only for the case of cookie-cutter sensors:
  - the searcher should more or less go back and forth across the channel, and

$$PD \approx \min(1, \frac{W}{LU}\sqrt{U^2 + V^2})$$



NAVAL POSTGRADUATE SCHOOL

Search games

- There are three kinds of targets
  - those that want to be found (and rescued)
  - those that don't care (or don't know there is a search in progress)
  - those that don't want to be found (evasive targets, our main interest)
- Two-person, zero-sum game theory deals with evasive targets
  - rarely applied in World War Two (no theory and no computers)
- Example: the evader can hide in either box A or box B, his choice
  - you can look in either box, but not both boxes
  - if the evader is in box A, and you look there, the detection probability is 0.5
  - if the evader is in box B, and you look there, the detection probability is 0.25
  - where should you look? Ans: (1/3, 2/3) in box (A,B)
- Other examples
  - the Princess and the Monster game of motion in a disk has been solved, instructively
  - the Flaming Datum problem, revisited
  - the search for Improvised Explosive Devices on roads
  - allocation of ASW forces to a large area

NAVAL POSTGRADUATE SCHOOL

References

- Hohzaki, R. and Washburn, A.R., “The Diesel Submarine Flaming Datum Problem,” *Military Operations Research*, Vol. 6, No. 4, 2001, pp. 19-30.
- ICAO, 2003. International Aeronautical and Maritime Search and Rescue Manual, volume 2, International Civil Aviation Organization document 9731-AN/958, appendix N, published jointly with the International Maritime Organization.
- OEG, 1946. Search and Screening, Operations Evaluation Group report 56, Navy department, Washington, DC.
- Soto, A. “The Flaming Datum Problem with Varying Speed”, Naval Postgraduate School Master’s thesis, 2000.
- Stone, L. Theory of Optimal Search, Academic Press, New York, 1975.
- Wagner, D.; Mylander, W.; and Sanders, T., *Naval Operations Analysis (third ed.),* Naval Institute Press, 1999.
- Washburn, A. *Search and Detection (fourth ed.),* INFORMS Topics in Analysis series, 2002.
- Washburn, A. and Kress, M., *Combat Models*, Springer, 2009.